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The Fixed Point Property in Banach Spaces with the NUS-Property

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In this paper, we show that the weak nearly uniform smooth Banach spaces have the fixed point property for nonexpansive mappings. © 1997 Academic Press

1. INTRODUCTION

Let C be a nonempty bounded closed convex subset of a Banach space X. A mapping $T: C \to C$ is said to be nonexpansive whenever the inequality $\|Tx - Ty\| \leq \|x - y\|$ holds for every $x, y \in C$.

We will say that X has the weak fixed point property (FPP) if every nonexpansive mapping $T: K \to K$, where K is a weakly compact convex subset of X, has a fixed point.

Since 1965, R. Browder, D. Gohde, W. A. Kirk (see [13]), and other authors have established that, under various conditions of a geometric kind on the norm of X, the FPP is guaranteed. Uniform convexity and normal structure are examples of such conditions.

Although it is well known that the FPP is not a self-dual property, $(l_1, \|\cdot\|_1)$ has the FPP but $(l_\infty, \|\cdot\|_\infty)$ lacks it, a classical result by Turett [20] states that if the characteristic of convexity of X, $\epsilon_0(X)$, is less than 1, then both X and X^* are superreflexive spaces with normal structure. In this way, Kutzurova, Maluta, and Prus [14] have proved that whenever X has property (β) of Rolewicz then both X and X^* have normal structure.

Among properties weaker than property (β), one of the most extensively examined is that of nearly uniform convexity (NUC) [9], which is

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known to imply normal structure. In [17] Prus gave a characterization of Banach spaces which are duals of NUC spaces. He called such spaces nearly uniform smooth (NUS). Bynum's space [2] provided an example of a NUC-space whose dual lacks normal structure, therefore it was an open question whether Banach spaces which are NUS do, or do not, have the FPP.

It is also still unknown whether uniformly nonsquare Banach spaces have the FPP (see [5, 11, 19] for partial affirmative results).

Recently, the author [6] has given a characterization of WNUS Banach spaces, via a Banach space coefficient. The notion of WNUS is a natural generalization of the property NUS.

In this paper, we show that WUNS, and hence NUS, Banach spaces have the FPP thereby answering the above question; moreover we will see that some uniformly nonsquare Banach spaces enjoy the FPP.

2. DEFINITIONS AND PRELIMINARIES

The following definitions, properties, and results will be used in this paper.

Let $\epsilon_0(X)$ be the characteristic of convexity of X defined by $\epsilon_0(X) = \sup\{\epsilon \in [0, 2] : \delta(\epsilon) = 0\}$, where $\delta(\epsilon) = \inf\{1 - \frac{1}{2}\|x + y\| : \|x\| \leq 1,\ \|y\| \leq 1,\ \|x - y\| \geq \epsilon\}$ is the modulus of convexity of X. Recall that a Banach space X is said to be uniformly nonsquare whenever $\epsilon_0(X) < 2$.

Let X be a Banach space with a Schauder basis (e_n). Thus any element $x \in X$ has a unique representation

$$x = \sum_{n=1}^{\infty} x_n e_n$$

and can be identified with "the coordinate sequence" $\{x_n\}$. Let F be a subset of N. The standard projection of X onto F is defined by

$$P_F(x) = \sum_{n \in F} x_n e_n.$$

When we write P_i we mean the natural projection of X onto the subspace spanned by $\{e_1, \dots, e_i\}$.

A Banach space X has an unconditional Schauder basis (e_n), if (e_n) is a Schauder basis and moreover the following constant

$$\lambda = \sup\left\{ \left\| \sum_{i=1}^{\infty} \epsilon_i a_i e_i \right\| : \left\| \sum_{i=1}^{\infty} a_i e_i \right\| = 1,\ \epsilon_i = \pm 1 \right\}$$

is finite. In this case λ is called the unconditional basis constant for (e_n) or, equivalently, (e_n) is said to be λ-unconditional.

A Banach space X is NUS provided that for every $\epsilon > 0$ there is $\mu > 0$ such that if $0 < t < \mu$ and (x_n) is a basic sequence in the unit ball of X, then there exists $k > 1$ so that $\|x_1 + tx_k\| \leq 1 + t\epsilon$.

A natural generalization of this notion is WNUS.

A Banach space X is WNUS whenever it satisfies the above condition with "for every $\epsilon > 0$" replaced by "for some $\epsilon \in]0, 1[$."

Let X be a Banach space. We define the coefficient $R(X)$ by

$$R(X) = \sup\Bigl(\liminf_{n\to\infty} \|x_n + x\|\Bigr),$$

where the supremum is taken over all weakly null sequences (x_n) of the unit ball and all points x of the unit ball.

THEOREM (A) (see [6]). *Let X be a Banach space. The following conditions are equivalent*:

(a) *X is WNUS*

(b) *X is reflexive and $R(X) < 2$.*

Let X be a Banach space. We denote by $[X]$ the quotient space $l_\infty(X)/c_0(X)$ endowed with the quotient norm given by $\|[z_n]\| := \limsup_{n\to\infty} \|z_n\|$, where $[z_n]$ denotes the equivalent class of $(z_n) \in l_\infty(X)$.

For $x \in X$ $(M \subset X)$, when we write $x \in [X]$ $(M \subset [X])$ we understand it to mean $x = [(x, x, \dots)] \in [X]$ $(M = \{x \in [X] : x \in M\})$. When K is a subset of X, we can consider the set $[K] := \{[z_n] \in [X] : z_n \in K, n = 1, 2, \dots\}$.

Suppose that C is a weakly compact convex subset of a Banach space X, and $T : C \to C$ is a nonexpansive mapping. The set C contains a weakly compact convex subset K which is minimal for T. That means $T(K)$ is contained in K and no strictly smaller weakly compact convex subset of K is invariant under T. If K contains only one point then T has a fixed point. Otherwise we can assume that $\operatorname{diam}(K) > 0$. It is easy to see that K contains a sequence (x_n) with $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$ (such a sequence is called an approximate fixed point sequence (a.f.p.s.) for T). We can define the mapping $[T] : [K] \to [K]$ by $[T]([x_n]) = [(Tx_n)]$.

A well known property of minimal sets is the following Goebel–Karlovitz lemma: (see [7, 12]).

LEMMA [GK]. *Let K be a minimal weakly compact convex subset for a nonexpansive mapping T, and let (x_n) be an a.f.p. sequence for T. Then for all $x \in K$*

$$\lim_{n\to\infty} \|x_n - x\| = \operatorname{diam}(K).$$

The next lemma is a slight modification of the Goebel–Karlovitz result (see [18, 4, 16]).

LEMMA [Lin]. *Let* $[W]$ *be any nonempty closed convex subset of* $[K]$ *which is invariant under* $[T]$. *Then* $\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = \operatorname{diam}(K)$ *for every* $x \in K$.

3. FIXED POINT RESULTS

Let X be a Banach space. If $(x_n) \in l_\infty(X)$ and $(y_n) \in c_0(X)$, then

$$D(x_n) = D(x_n + y_n),$$

where $D(x_n) := \limsup_{n\to\infty} \limsup_{m\to\infty} \|x_n - x_m\|$.

This fact allows us to give the following definition:

DEFINITION. Let X be a Banach space and consider as in Section 2 the space $[X]$. If $[x_n] \in [X]$, then $D([x_n]) := D(x_n)$.

The following result generalizes Corollary 2.4 of [6], in the sense that the weak Opial condition can be omitted.

THEOREM 3. *Let X be a Banach space such that $R(X) < 2$. Then X has the FPP.*

Proof. To get a contradiction, suppose that X fails to have the FPP. Then there exists a weakly compact convex subset K of X such that $\operatorname{diam}(K) = 1$, which is minimal for a nonexpansive mapping $T : K \to K$. Without loss of generality we can assume that there exists a weakly null a.f.p. sequence (x_n) for T in K.

We consider the following subset of $[X]$:

$$[W] := \{[z_n] \in [K] : \|[z_n] - [x_n]\| \le 1/2, D([z_n]) \le 1/2\}.$$

It is easy to see that $[W]$ is a closed set of $[X]$ and moreover it is clear that $[W]$ is convex and $[T]$-invariant subset of $[K]$.

$[W]$ is nonempty since by Lemma [GK] it is not difficult to see that $[x_n/2] \in [W]$.

Thus by Lemma [Lin] we see that

$$\sup\{\|[w_n] - x\| : [w_n] \in [W]\} = 1$$

for all $x \in K$.

Let $[z_n]$ be an element of $[W]$.

$$\|[z_n]\| = \limsup_{n\to\infty} \|z_n\| = \lim_{k\to\infty} \|z_{n_k}\|,$$

for some subsequence (z_{n_k}) of (z_n). As K is a weakly compact subset of X we can suppose that (z_{n_k}) is weakly convergent to $y \in K$. Then, passing to subsequences and using a diagonal argument, we can assume that for each $k \in N$

$$\left(1 - \frac{1}{k}\right)\|z_{n_k} - y\| \leq \liminf_{n\to\infty} \|z_{n_k} - y\|.$$

For all $k \in N$, we define

$$y_k = \left(1 - \frac{1}{k}\right) \frac{z_{n_k} - y}{\max\left\{\liminf_{k\to\infty} \|z_{n_k} - y\|, \|y\|\right\}}.$$

Since (y_k) is a weakly null sequence of the unit ball, by the definition of the coefficient $R(X)$, we have that

$$\liminf_{k\to\infty} \left\| y_k + \frac{y}{\max\left\{\liminf_{k\to\infty} \|z_{n_k} - y\|, \|y\|\right\}} \right\| \leq R(X).$$

Consequently

$$\lim_{k\to\infty} \|z_{n_k}\| \leq R(X) \max\left\{ \liminf_{k\to\infty} \|z_{n_k} - y\|, \|y\| \right\}.$$

Now we are going to show that

$$\max\left\{ \liminf_{k\to\infty} \|z_{n_k} - y\|, \|y\| \right\} \leq 1/2.$$

Indeed, $z_{n_k} - x_{n_k} \rightharpoonup y$ and so

$$\|y\| \leq \liminf_{k\to\infty} \|z_{n_k} - x_{n_k}\| \leq \limsup_{n\to\infty} \|z_n - x_n\| = \|[z_n] - [x_n]\| \leq 1/2.$$

On the other hand, since $\text{w-lim}_{s\to\infty}(z_{n_k} - y - (z_{n_s} - y)) = z_{n_k} - y$, we have

$$\|z_{n_k} - y\| \leq \limsup_{s\to\infty} \|z_{n_k} - y - (z_{n_s} - y)\|.$$

Thus we derive

$$\limsup_{k\to\infty}\|z_{n_k}-y\|\le \limsup_{k\to\infty}\Big(\limsup_{s\to\infty}\|z_{n_k}-y-(z_{n_s}-y)\|\Big).$$

Since $D(z_{n_k}-y)=D(z_{n_k})\le D(z_n)$ and $D([z_n])\le 1/2$ we have that

$$\limsup_{k\to\infty}\|z_{n_k}-y\|\le D([z_n])\le 1/2.$$

The fact that

$$\max\Big\{\liminf_{k\to\infty}\|z_{n_k}-y\|,\|y\|\Big\}\le 1/2$$

implies

$$\|[z_n]-0\|=\lim_{k\to\infty}\|z_{n_k}\|\le R(X)/2<1$$

and this contradicts Lemma [Lin].

The following result is a consequence of the previous result and of Theorem (2.3) of [6].

COROLLARY 4. *A Banach space X has the FPP if there exists a Banach space Y such that $d(X,Y)R(Y)<2$, where $d(X,Y)$ is the Banach–Mazur distance between X and Y.*

A consequence of Theorem 3 is that NUS-Banach spaces enjoy the FPP. Indeed,

COROLLARY 5. *If X is a WNUS Banach space, then X has the FPP. In particular NUS Banach spaces have the FPP.*

Proof. If X is a WNUS-space, then by Theorem [A],

$$R(X)<2.$$

Thus, by Theorem 3, we derive that X has the FPP.

In [6] the author proved that, if X is a reflexive WUKK'-space, then X^* is a WNUS-space.

Recall that a Banach space X is said to be a WUKK'-space if there exists $\epsilon\in]0,1[$ and $\delta>0$ such that $\|x\|\le 1-\delta$ whenever x is a weak limit of some sequence (x_n) in the unit ball with $\liminf_{n\to\infty}\|x_n-x\|\ge\epsilon$ (see [15]).

It is not difficult to see that a Banach space X is WUKK' whenever X is NUC.

COROLLARY 6. *If X is a reflexive WUKK'-space, then X and X^* have the FPP.*

4. BANACH SPACES WITH $R(X) < 2$

Following the ideas in the definition of a nearly uniform smooth space by S. Prus and of the modulus of uniform smoothness, Dominguez-Benavides [3] defines a modulus of nearly uniform smoothness as follows.

Let X be a Banach space. The modulus of nearly uniform smoothness of X is the function

$$\Gamma_X(t) = \sup\left\{\inf\left(\frac{\|x_1 + tx_n\| + \|x_1 - tx_n\|}{2} - 1 : n > 1\right)\right\},$$

where the supremum is taken over all basic sequences (x_n) of the unit ball.

By using this coefficient Dominguez-Benavides showed that a Banach space X is NUS if and only if X is reflexive and $\lim_{t\to 0}(\Gamma_X(t)/t) = 0$.

A open question which is posed in [3] is: Does $\lim_{t\to 0}(\Gamma_X(t)/t) < 1/2$ imply the FPP?

It is not difficult to see that if X is a reflexive Banach space with $\lim_{t\to 0}(\Gamma_X(t)/t) < 1/2$, then X is WUNS and therefore $R(X) < 2$, thus by Theorem 3, X has the FPP.

On the other hand, it is easy to show that if X is a Banach space such that $\epsilon_0(X) < 1$ then $R(X) < 2$; however, when $\epsilon_0(X) \geq 1$ it is unknown whether X has or does not have the FPP. A partial answer is given in [5] by showing that those uniformly nonsquare Banach spaces which satisfy the WORTH-property have the FPP. The WORTH-property was introduced by B. Sims in [18] as follows: a Banach space X has WORTH-property if

$$\lim_{n\to\infty} \big| \|x_n - x\| - \|x_n + x\| \big| = 0$$

for all $x \in X$ and for all weakly null sequence (x_n). It seems to be unknown whether the WORTH-property implies FPP.

In [19] the author measures the "degree of WORTHness" of a Banach space X by

$$w = \sup\left\{r > 0 : r \liminf_{n\to\infty} \|x + x_n\| \leq \liminf_{n\to\infty} \|x - x_n\|\right\},$$

where the infimum is taken over all $x \in X$ and over all weakly null sequences (x_n) in X (so X has the WORTH-property if and only if $w = 1$), and thus B. Sims obtains the following.

PROPOSITION 8 OF [19]. *A Banach space X has weak normal structure whenever $w > \max\{\epsilon/2, 1 - \delta(\epsilon)\}$ for some positive ϵ.*

In [11] Jiménez-Melado and Llorens-Fuster gave a sufficient condition for FPP by using the reciprical coefficient of w, which can be rewritten as

$$\mu = \frac{1}{w} = \inf\Big\{ r > 0 : \liminf_{n \to \infty} \|x_n + x\| \leq r \limsup_{n \to \infty} \|x_n - x\| \text{ whenever } x_n \rightharpoonup 0 \text{ and } x \in X \Big\}.$$

THEOREM 1 OF [11]. *A Banach space X has the FPP whenever* $\epsilon_0/4 + \mu/2 < 1$.

In the next result we give a more general affirmative answer in this direction.

THEOREM 7. *Let X be a Banach space. If* $\mu\epsilon_0(X) < 2$ *then* $R(X) < 2$.

Proof. Since $\mu\epsilon_0(X) < 2$ then we can select $\tau > 0$ such that $\epsilon_0(X)(1 + \tau)\mu < 2$.

Let (x_n) be a weakly null sequence in the unit ball and let $x \in B_X$. To get the result we must compute

$$\liminf_{n \to \infty} \|x_n + x\|.$$

If the last limit is bigger than $\epsilon_0(X)(1 + \tau)\mu$, then by the definition of μ we have

$$\epsilon_0(X)(1 + \tau)\mu < \liminf_{n \to \infty} \|x_n + x\| \leq \mu \limsup_{n \to \infty} \|x_n - x\|,$$

therefore, there exists a subsequence (x_{n_k}) of (x_n) such that for all $k \in N$

$$\epsilon_0(X)(1 + \tau) < \|x_{n_k} - x\|.$$

The subsequence (x_{n_k}) and x satisfy

(a) $\|x_{n_k}\| \leq 1$ for all $k \in N$,

(b) $\|x\| \leq 1$,

(c) $\|x_{n_k} - x\| > \epsilon_1 := \epsilon_0(X)(1 + \tau) > \epsilon_0(X)$ for all $k \in N$.

Hence, by using the properties of the modulus of convexity, we derive

$$\liminf_{k \to \infty} \|x_{n_k} + x\| \leq 2\big(1 - \delta(\epsilon_1)\big),$$

which means that if (x_n) is a weakly null sequence of the unit ball and $x \in B_X$ then

$$\liminf_{n \to \infty} \|x_n + x\| \leq \max\big\{\epsilon_0(X)\mu(1 + \tau), 2\big(1 - \delta(\epsilon_1)\big)\big\} < 2$$

and thus we get that $R(X) < 2$.

COROLLARY 8. *Every Banach space X such that $\epsilon_0 \mu < 2$ has the FPP and it is WNUS.*

Remark. Corollary 8 is a generalization of Theorem 1 of [11] since the inequality $\epsilon_0 \mu < 2$ holds when $\epsilon_0/4 + \mu/2 < 1$.

If X is a Banach space with a λ-unconditional basis, Jiménez-Melado and Llorens-Fuster in Corollary 1 of [11] show that $\lambda \leq \mu$ and hence if $\epsilon_0 \lambda < 2$ we derive that $R(X) < 2$. It is well known that there exist Banach spaces with $\epsilon_0 = 1$ without normal structure (see [8]); for these spaces, the above theorem yields X has the FPP whether $\lambda < 2$. Proposition 8 of [19] does not recapture this since these spaces fail to have normal structure.

In [10] the authors give a condition on a general Banach space which implies the FPP for nonexpansive mappings; they call the property O.C. To test whether a given Banach space is O.C. is not an easy task. However, for Banach spaces having a Schauder basis (e_n) it is possible to define a coefficient which allows us to easily identify a wide class of O.C. Banach spaces.

The coefficient $\rho(X)$ associated to (e_n) is

$$\rho(X) := \sup\left\{\|x\| : \exists i \in N \text{ with } \|P_i(x)\| \leq 1, \|(I - P_i)(x)\| \leq 1\right\}.$$

Let us remark that for $1 \leq p < \infty$ it holds that $\rho(l_p) = 2^{1/p}$ and if E_β is the space l_2 endowed with the norm

$$|x|_\beta = \max\{\|x\|_2, \beta\|x\|_\infty\}$$

then $\rho(E_\beta) = \sqrt{2}$. W. L. Bynum [2] defined the Banach space $l_{p,\infty}$ as the space l_p renormed according to

$$|x| = \max\left\{\|x^+\|_p, \|x^{-1}\|_p\right\},$$

where x^+ and x^- are respectively the positive and negative part of x. The space $l_{p,\infty}$ lacks asymptotic normal structure but $\rho(l_{p,\infty}) = 2^{1/p}$.

By using $\rho(X)$ Jiménez-Melado and Llorens-Fuster in [10] obtain the following result.

Let X be a Banach space with a Schauder basis (e_n) such that $\|P_i\| = \|I - P_i\| = 1$ for all $i \in N$. If $\rho(X) < 2$ then X is O.C. and thus X has the FPP.

The following proposition gives a relationship between $\rho(X)$ and $R(X)$.

PROPOSITION 9. *Let X be a Banach space with a Schauder basis (e_n). Then $R(X) \leq \rho(X)$.*

Proof. Let (x_n) be a weakly null sequence in the unit ball and let $x \in B_X$.

Consider $M = \sup\{\|I - P_n\|, \|P_n\| : n \in N\}$. It is clear that $M \geq 1$.

Given $\epsilon > 0$, there exists $n_0 \in N$ such that $\|(I - P_{n_0})(x)\| \leq \epsilon/3M$ and hence, $\|P_{n_0}(x)\| \leq \|x\| + \epsilon/3M$.

On the other hand, since (x_n) is a weakly null sequence by the Bessaga–Pelzynski theorem, there exists a subsequence $(x_{\beta(n)})$ of (x_n) satisfying

$$\|P_{[a_n, b_n]}(x_{\beta(n)}) - x_{\beta(n)}\| \leq \frac{\epsilon}{3M},$$

where (a_n) and (b_n) are increasing sequences of natural numbers with $a_{n+1} > b_n > a_n$ for all $n \in N$.

For all $n \in N$ with $n > n_0$ we have

$$\|(I - P_{n_0})(x_{\beta(n)} - x)\| \leq M\|x_{\beta(n)} - P_{[a_n, b_n]}(x_{\beta(n)})\| \\ + \|P_{[a_n, b_n]}(x_{\beta(n)})\| + M\|x - P_{n_0}(x)\|.$$

Since $M \geq 1$ and $\|P_{[a_n, b_n]}(x_{\beta(n)}) - x_{\beta(n)}\| \leq \epsilon/3M$ we derive

$$\|(I - P_{n_0})(x_{\beta(n)} - x)\| \leq 1 + \epsilon. \qquad (*)$$

On the other hand, it is easy to see that if $n > n_0$ then

$$\|P_{n_0}(x_{\beta(n)} + x)\| \leq 1 + \epsilon. \qquad (**)$$

Therefore, by $(*)$, $(**)$ and the definition of $\rho(X)$ we obtain that for all $n > n_0$,

$$\|x_{\beta(n)} + x\| \leq (1 + \epsilon)\rho(X)$$

hence

$$\liminf_{n \to \infty}\|x_n + x\| \leq \liminf_{n \to \infty}\|x_{\beta(n)} + x\| \leq (1 + \epsilon)\rho(X)$$

which implies that $R(X) \leq \rho(X)$.

Remark. The coefficient $R(X)$ can become strictly less than $\rho(X)$, since, for example, $\rho(l_1) = 2$ while $R(l_1) = 1$.

As a consequence of the last proposition we obtain a generalization of the above result of Jiménez-Melado and Llorens-Fuster in the sense that the condition $\|P_i\| = \|I - P_i\| = 1$ for all $i \in N$ can be omitted.

COROLLARY 10. *If X is a Banach space with $\rho(X) < 2$, then X has the FPP.*

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